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NORSIG '98
IEEE Nordic Signal Processing Symposium
Vigsø, Denmark
June 8-11, 1998

ON OUTPUT DISTRIBUTIONS OF RATIONAL FILTERS IN THE CASE OF UNIFORM NOISE

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ABSTRACT

This paper illustrates the difficulties in the evaluation of the output distributions of rational filters even when the assumptions of the relation between the signal and the noise are very simplified. We derive the output distribution analytically for one of the simplest rational filter for two i.i.d. uniformly distributed input samples. From this derivation the inherent difficulties can be understood.

1. INDRODUCTION

The output distribution of a filter for given statistical assumptions on the relation between the signal and the noise give a lot of valuable information on the behavior of the filter. First of all, the mean value of the distribution can be used to verify is the filter unbiased for the corresponding situation. Unbiasness is an important concept in signal homogeneous regions, where it is desirable that the expected value of the output of the filter is that of the signal value in the region, i.e., filter does not cause bias. The second order central output moment is another important numerical measure that can be calculated from the output distribution. It is often used to measure the noise attenuation capability of the filter because it quantifies the spread of the output samples with respect to their mean value. Other numerical fidelity measures of the filter can be derive from the output distribution as well.

The rational filters are used for instance in image processing with a view to effectively attenuate the noise that corrupts an image while introducing small distortions on the image details. The rational filters are described by rational functions, i.e., by the ratio of two polynomials in the input variables. An effective rational filter tested in practice is introduced by G. Ramponi: [1]

$$y_n = \frac{w(x_{n-1}+x_{n+1})}{wk(x_{n-1}-x_{n+1})^2+1} + (1 - \frac{2w}{wk(x_{n-1}-x_{n+1})^2+1})x_n. \quad (1)$$

In this paper the distribution of rational filtered random variable is solved analytically in the special case where the rational filter is the simpliest possible and the input signal is independently (0,1) uniformly distributed.

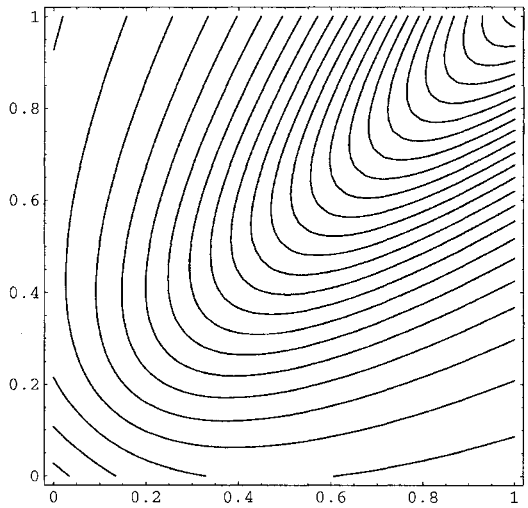

Figure 1: A contour plot of (2) for parameter values $a_1 = 0.45$, $a_2 = 0.55$ and $b = 5$

2. ADDRESSED PROBLEM

Let X and Y be two independent (0,1) uniformly distributed random variables. The problem is to solve the distribution of the random variable U defined as:

$$U = \frac{a_1 X + a_2 Y}{1 + b(X-Y)^2}, \quad (2)$$

where a_1, a_2 and b are some nonnegative constants.

3. SOLUTION

To obtain the cumulative distribution function $F(U)$ we calculate the probability $P(U \leq u)$ for values of u from 0 to $a_1 + a_2$. For a fixed value of u $F(u)$ equals to the area restricted by the slope $\frac{a_1 x + a_2 y}{1+b(x-y)^2} \leq u$ in the square with corners (0,0), (0,1), (1,0), and (1,1) in the xy-space.

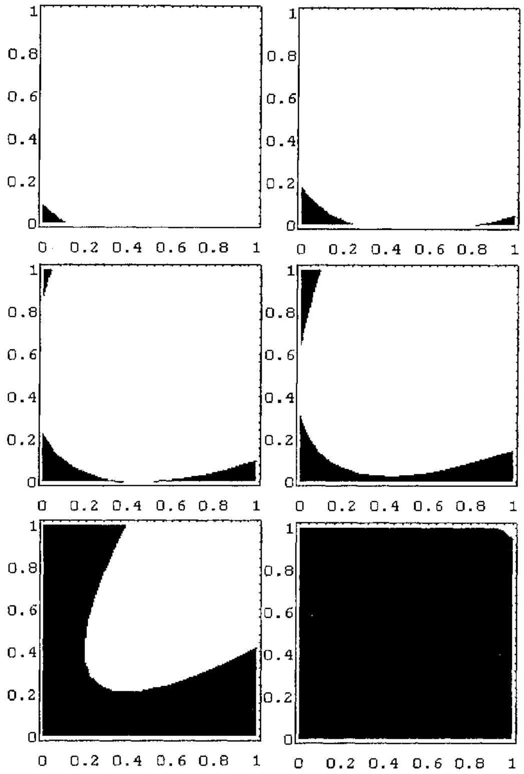

Figure 2: An example how the area to be integrated grows from different corners

The idea is simple but in practice there are many ways how the slope can cut the borders of the square. In Figure 1 a contour plot of (2) is presented for parameter values $a_1 = 0.45$, $a_2 = 0.55$ and $b = 5$. The slope can be modelled by two function $g_1(x, u)$ and $g_2(x, u)$ whose expressions can be solved from (2).

Figure 2 illustrate how the area to be integrated grows from different corners. Unfortunately the order is not always the same; for example the upper left corner can come before than the right lower corner. The order of critical values of u i.e., where significant charge in the integration area occurs depends on parameters. Two first critical values of u, $\frac{a_1}{1+b}$ and $\frac{a_2}{1+b}$ are obtained by substituting the coordinates of corners (1,0) and (0,1) to equation (2). Two second critical values of u, $\frac{a_2}{2\sqrt{b}}$ and $\frac{a_1}{2\sqrt{b}}$ are obtained from (2) by setting $x = 0$ and $y = 0$ and finding u so that the equation has only one solution. The critical value $u_{AO} = \frac{4b(a_1+a_2)+\sqrt{-16a_2(2a_1+a_2)+16b^2(a_1+a_2)^2}}{8b}$ is such a value of u that the cutting point of $g_1(x, u)$ and $g_2(x, u)$ is on the line $y = 1$. Accordingly what values parameters a_1, a_2 and b have, there are five different cases:

1. $b < 1$ and $a_2 < \frac{a_1(1+b)}{2\sqrt{b}}$
2. $b \geq 1$ and $a_1 \geq a_2$
3. $b \geq 1$ and $a_1 < a_2 < \frac{a_1(1+b)}{2\sqrt{b}}$
4. $b \geq 1$ and $a_2 \geq \frac{a_1(1+b)}{2\sqrt{b}}$
5. $b < 1$ and $a_2 \geq \frac{a_1(1+b)}{2\sqrt{b}}$

For the special case 3, where $b \geq 1$ and $a_1 \leq a_2 < \frac{a_1(1+b)}{2\sqrt{2}}$,the cumulative distribution function $F(u)$ is:

$$F(u) = \begin{cases} F_{31}(u) & ,0 \leq u < \frac{a_1}{1+b} \\ F_{32}(u) & ,\frac{a_1}{1+b} \leq u < \frac{a_2}{1+b} \\ F_{33}(u) & ,\frac{a_2}{1+b} \leq u < \frac{a_1}{2\sqrt{b}} \\ F_{34}(u) & ,\frac{a_1}{2\sqrt{b}} \leq u < \frac{a_2}{2\sqrt{b}} \\ F_{35}(u) & ,\frac{a_2}{2\sqrt{b}} \leq u < u_{AO} \\ F_{36}(u) & ,u_{AO} \leq u < a_1 + a_2, \end{cases} \tag{3}$$

where
$u_{AO} = \frac{4b(a_1+a_2)+\sqrt{-16a_2(2a_1+a_2)+16b^2(a_1+a_2)^2}}{8b}$.
The expressions of functions $F_{31}, \ldots, F_{36}$ consist of square roots and second order terms. These functions can be presented following way

$$\begin{aligned} F_{31}(u) &= \sqrt{S_2 + 2(a_1 + a_2)S_1}(Q - \frac{S_1}{6b^2u^2}) + \\ & \quad \sqrt{S_2}Q + \frac{a_2 S_1}{4b^2u^2} \\ F_{32}(u) &= \frac{1}{2} + \frac{a_2}{2bu} + \sqrt{S_2}Q + R_2(-\frac{1}{3bu} + Q) \\ F_{33}(u) &= \frac{1}{2} + \frac{a_2}{2bu} + 2\sqrt{S_2}Q\frac{a_2 + 2bu + \frac{1}{2}}{4b^2u^2} + \\ & \quad \sqrt{S_2 + 2(a_1 + a_2)R_1}(Q + \frac{S_1}{6b^2u^2}) \\ F_{34}(u) &= F_{33}(u) \\ F_{35}(u) &= F_{33}(u) - 2\sqrt{S_2}Q \\ F_{36}(u) &= F_{35}(u), \end{aligned}$$

where
$S_1 = a_1 - \sqrt{a_1^2 - 4bu^2}$
$S_2 = a_2^2 - 4bu^2$
$R_1 = a_1 + 2bu - \sqrt{a_1^2 + 4a_1bu - 4bu^2}$
$R_2 = \sqrt{a_2^2 - 4a_1bu + 4a_2bu - 4bu^2}$
$Q = -S_2\frac{1}{12(a_1+a_2)b^2u^2}$.

In Figure 3 the cumulative distribution function is presented for parameter values $a_1 = 0.45$, $a_2 = 0.55$ and $b = 5$. In Figure 4 the corresponding density function is presented. For these parameter values the expected value of value of the output of the filter is 0.22 and the variance of the output 0.04. In Figures 5, 6 and 7 the density functions are presented for cases where $a_1 = 0.45$, $a_2 = 0.55$ and b have values 10, 20 and 30, respectively. From these figures it is easy to observe that the expected value and the variance of the output decrease when b increases.

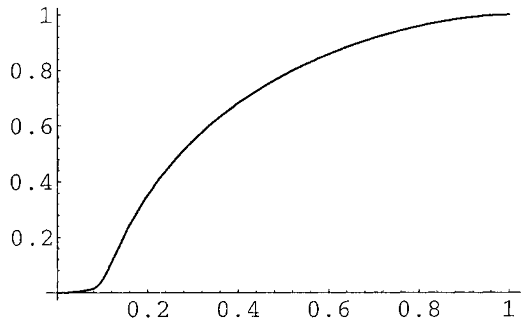

Figure 3: The cumulative distribution function for parameter values $a_1 = 0.45$, $a_2 = 0.55$ and $b = 5$

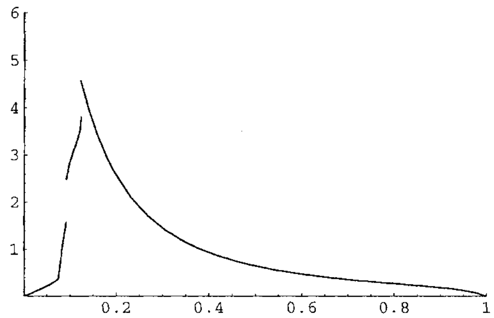

Figure 4: The density function for parameter values $a_1 = 0.45$, $a_2 = 0.55$ and $b = 5$

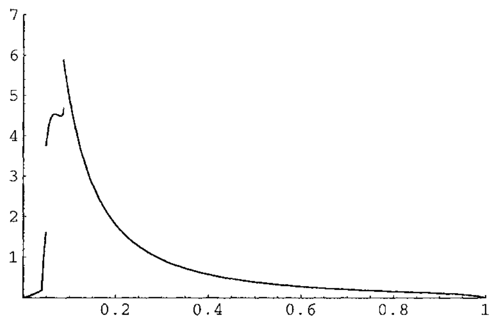

Figure 5: The density function for parameter values $a_1 = 0.45$, $a_2 = 0.55$ and $b = 10$

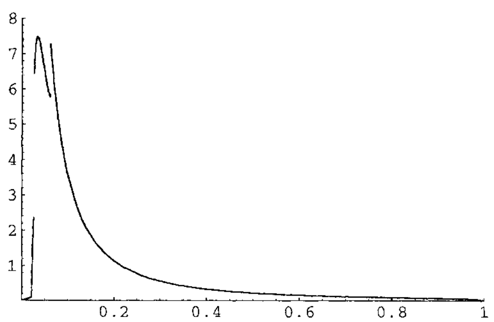

Figure 6: The density function for parameter values $a_1 = 0.45$, $a_2 = 0.55$ and $b = 20$

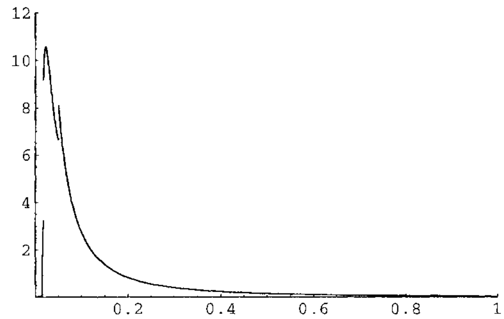

Figure 7: The density function for parameter values $a_1 = 0.45$, $a_2 = 0.55$ and $b = 30$

4. CONCLUSIONS

In this paper the distribution of signal filtered by a simple rational filter is solved analytically in the case where the input signals are (0,1) uniform. Although the rational filter is simple, the resulting cumulative distribution function is not simple at all. It is possible to solve the cumulative distribution function for other rational filters in a similar way but as illustrated above technical difficulties are expected. These results motivate the efforts to find good approximation methods to obtain the output distribution in a more convenient manner.

5. REFERENCES

[1] Giovanni Ramponi, "The Rational Filter for Image Smoothing," *IEEE Signal Processing Letters*, vol. 3, no. 3, pp. 63-65, March 1996 .